ON π-EXTENSIONS OF THE SEMIGROUP $\mathbb{Z}_+$

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Abstract. We study inverse π-extensions of the semigroup $\mathbb{Z}_+$. It is shown that π-extension of the semigroup $\mathbb{Z}_+$ is inverse, iff its π-extension coincides with $\pi(\mathbb{Z}_+)$. The existence of a non-inverse π-extension for semigroup $\mathbb{Z}_+$ is proved.

Key words: inverse semigroup, inverse representation, Toeplitz algebra, π-extension, inverse π-extension, C^*-algebra.

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1. Introduction. In his well known work [1] Coburn proved that all the isometric representations of the semigroup $\mathbb{Z}_+$ of non-negative integers generate canonically isomorphic C^*-algebras. This theorem was generalized by many authors to a larger class of semigroups. Douglas [2] showed the same for the semigroup of positive cone of real numbers $\mathbb{R}$. Murphy proved this theorem for the positive cones of abelian groups with order. On the other hand, Murphy [3] and Jang [4] have shown that this theorem is not true for the semigroup $\mathbb{Z}_+\backslash\{1\}$. The isometric representations with commuting range projections of the semigroup $\mathbb{Z}_+\backslash\{1\}$ has been studied by Raeburn and Vittadello [5].

We introduce the notion of π-extension of the semigroup of non-negative integers $\mathbb{Z}_+$ (see definition 2.1), and study the properties of π-extensions of the semigroup $\mathbb{Z}_+$. Also the concept of the inverse π-extension of the semigroup $\mathbb{Z}_+$ is introduced in definition 5.1. We prove, that if π is an irreducible representation, then there is no non-trivial inverse π-extension for this semigroup. In case π is reducible, there exists a non-inverse π-extensions. On the other hand we show that for any isometric representation of $\mathbb{Z}_+$ there always exists π-extension.

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2. Preliminaries. Consider an isometric representation of the semigroup $\mathbb{Z}_+$:

$$\pi : \mathbb{Z}_+ \to B(H),$$

where $B(H)$ is a set of all bounded linear operators on Hilbert space H. Denote by $Is(H)$ the semigroup of all isometric operators in $B(H)$.

Definition 2.1. We call $M \subset Is(H)$ a π-extension of the semigroup $\mathbb{Z}_+$, if:
1. $\pi(\mathbb{Z}_+) \subset M$;
2. $\pi(i)T = T\pi(i)$ for T in M and i in $\mathbb{Z}_+$.

The following irreducible representation is considered throughout this paper, unless the opposite is mentioned:

$$\pi : \mathbb{Z}_+ \to B(H^2),$$

where $H^2(S^1, d\mu)$ is the Hardy space of square-integrable complex-valued functions on the unit circle S^1 by Haar measure μ, and with spectrum in $\mathbb{Z}_+$.

The operator $\pi(n)$ is the multiplicative operator of multiplication by the function $e^{in\theta}$, i.e.:

$$\pi(n) : H^2 \to H^2 \text{ and } \pi(n)f(e^{i\theta}) = e^{in\theta}f(e^{i\theta}).$$

The orthonormal system of functions 1, $e^{i\theta}$, $e^{i2\theta}$, $\ldots$ form a basis in H^2, and the operator $\pi(1)$ is the shift operator on this basis:

$$\pi(1)e^{in\theta} = e^{i(n+1)\theta}.$$

Therefore, the C^*-subalgebra of the algebra $B(H^2)$ generated by the operators $\pi(1)$ and $\pi^*(1)$ is a Toeplitz algebra.

3. Inverse Representations. *An inverse semigroup P* is a semigroup, such that each element x has a unique *inverse* element x^* satisfying

$$xx^*x = x, \ x^*xx^* = x^*.$$

We denote by Δ_S the set of all isometric representations of the semigroup S. For $\pi \in \Delta_S$ define S^π to be the semigroup generated by operators $\pi(i)$ and $\pi^*(i)$, where $i \in S$.

Definition 3.1. We call the representation $\pi \in \Delta_S$ inverse, if S^π is an inverse semigroup.

The regular isometric representation is a map $\pi : S \to B(l^2(S))$, $i \mapsto \pi(i)$, defined by following relation:

$$(\pi(i)f)(j) = \begin{cases} f(k), & \text{if } j = i + k \text{ for some } k \in S; \\ 0, & \text{otherwise.} \end{cases}$$

T h e o r e m 3.1. The regular isometric representation of the semigroup S is inverse (see [6, 7]).

Now we give an example of non-inverse representation.

Let $\pi : \mathbb{Z}_+ \to B(H^2)$ be the representation of the semigroup $\mathbb{Z}_+$ described in the section 2, i.e. $\pi(n)$ is the multiplicative operator of multiplication by the function $e^{in\theta}$.

Every inner function $\Phi(z)$ defines an isometric multiplicative operator T_Φ:

$$T_\Phi f = \Phi f.$$

T h e o r e m 3.2. Let $\widetilde{\pi} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to B(H^2)$ be a representation, which maps $(n, 0) \mapsto e^{in\theta}$ and $(0, m) \mapsto \Phi^m$, where Φ is an arbitrary inner function, not in semigroup $\{e^{in\theta}\}_{n=0}^{\infty}$. Then $\widetilde{\pi}$ is a non-inverse representation, i.e. $(\mathbb{Z}_+ \times \mathbb{Z}_+)^{\widetilde{\pi}}$ is a non-inverse semigroup.

4. π-extension of the semigroup $\mathbb{Z}_+$.

L e m m a 4.1. Every isometric operator in π-extension of the semigroup $\mathbb{Z}_+$ can be represented through a single inner function.

Let us denote by $C^*_\pi(\mathbb{Z}_+)$ the C^*-algebra generated by the isometric representation π, described in Section 2. Let also $C^*_\pi(M)$ be the C^*-algebra, generated by a semigroup $M \subset Is(H^2)$.

If M is a π-extension of the semigroup $\mathbb{Z}_+$, then by Lemma 4.1 for each isometric operator $T \in M$ there exists a unique inner function Φ, such that the operator T is a multiplication operator by Φ. Define

$$M' = \{\Phi; \ T_\Phi \in M\}.$$

T h e o r e m 4.1. Let M be the π-extension of the semigroup $\mathbb{Z}_+$. Then the following conditions are equivalent:

1. $C^*_\pi(\mathbb{Z}_+) = C^*_\pi(M)$;
2. $M^{'}$ is a subsemigroup of the semigroup of finite Blaschke products.

5. Inverse π-extension. We denote by $\mathbb{Z}_+^\pi$ the involutive semigroup generated by $\pi(\mathbb{Z}_+)$ and $\pi(\mathbb{Z}_+)^*$. Let M be the π-extension of the semigroup $\mathbb{Z}_+$. Denote by $\mathcal{M}^*$ the semigroup generated by M and M^*. *Definition 5.1.* We call the π-extension of the semigroup $\mathbb{Z}_+$ inverse, if $\mathcal{M}^*$ is an inverse semigroup.

Let $\pi : \mathbb{Z}_+ \to B(H^2)$ be the representation of the semigroup $\mathbb{Z}_+$, described in Section 2. Then the following result is true.

T h e o r e m 5.1. $\mathcal{M}^*$ is inverse iff $\mathcal{M}^* = \mathbb{Z}_+^\pi$.

Consider an arbitrary isometric representation $\pi : \mathbb{Z}_+ \to B(H)$. Denote $H_0 = \ker \pi^*(1)$. It is clear that H_0 is a Hilbert subspace of H.

T h e o r e m 5.2. Let $\pi : \mathbb{Z}_+ \to B(H)$ be an isometric representation of the semigroup $\mathbb{Z}_+$ such that the subspace $H_0 = \ker \pi^*(1)$ is not one dimensional. Then there exists an inverse π-extension M of the semigroup $\mathbb{Z}_+$ such that $\mathbb{Z}_+^\pi$ is a proper involutive subsemigroup of the involutive semigroup $\mathcal{M}^*$.

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